Exponentials in Physics

The exponential function crops up all over physics. In this module, we will look at why this is, see what an exponential means in terms of the underlying physics, and look at some practical examples of working with exponentials.

The exponential function

The exponential function is written as $\mathbf{e^x}$ or **exp(x)**, where e is an irrational number

e ≈ 2.71828.

The exponential acts as an 'inverse' of the natural logarithm (the logarithm to base e).

$$y = e^x \Leftrightarrow x = \log_e y$$

The exponential function can be expanded as: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

To differentiate the exponential function: $\frac{d}{dx}(e^{ax}) = ae^{ax}$

To integrate the exponential function: $\int e^{ax}dx = \frac{1}{a}e^{ax} + c$

Example 1: radioactive decay.

Consider a box containing a population of 1000 radioactive nuclei. In each time interval, Δt, each particle has a 1 in 10 chance of decaying. We cannot make a predication about an individual nucleus, but we can say how the population will change as a whole.

In the first time interval, if each particle has a 1 in 10 chance of decaying, we can say that 100 of the 1000 particles will decay. Fill in the table below to show how this process continues for 10 time intervals.

Time interval	Number at start	Decays	Number at end
1	1000	100	900
2	900	90	810
3	810	81	729
4	729	73	656
5	656	66	590
6	590	59	531
7	531	53	478
8	478	48	430
9	430	43	387
10	387	39	348

Plot your values with your plotting system of choice (excel, graph paper, whatever) and verify that the behaviour is roughly as you would expect… you should see a slow decline from the initial population.

Verify the plots are OK. Has anyone tried to plot error bars?

Now transfer your values from the table above into the first column of the table below. We will now try to generate a model (a mathematical description) of the behaviour of our population of radioactive particles to compare to the results.

From your plot, it should be obvious that the population change is not a linear one! In fact, the obvious model to try is that of exponential decay:

$$I = I_0 e^{-\lambda t} \qquad (1)$$

where I_0 is the initial population, λ is the decay constant describing the rate of decay of the population, and t is the elapsed time.

We can estimate the decay constant from the result of the first time interval:

When t = 1, I/I_0 = 0.9, and so $0.9 = e^{-\lambda} \Rightarrow \lambda = -\ln(0.9) = 0.10536$

Calculate the values of the model, and insert them in the table below:

Time interval	Number at end	Model
1	900	
2	810	
3	729	
4	656	
5	590	590.5
6	531	
7	478	
8	430	
9	387	
10	348	

Verify that the model is a good description of the situation.

It should be very close! There is absolutely nothing wrong with the model, and the 'empirical' values are only limited by the width of each time bin and maybe some rounding errors.

So far, we have used the decay constant, λ, to describe how fast the decay occurs, by modifying the speed at which the exponential affects the initial population. It is also common, especially for radioactive decay, to discuss the half-life of the decaying isotope. This is defined as the time after which one half of the population has decayed.

Use your graph to estimate the half-life for this population (in units of 'time intervals')

Again, this should come out pretty close, all is needed is to read off the point corresponding to 500 nuclei left, which should be about 6.5 time units (actually 6.57)

Looking at this more mathematically, there is a straightforward relationship between the half-life and the decay constant.

Use equation (1) above to get a formula for the half-life of the population in terms of the decay constant λ.

Half-life = ln2 / lambda ; obtained by just inserting values of $I=I_0/2$ and t = half-life in the formula above

Compare the half-life given by your formula above and your estimate from the graph.

The values should come out very close; make sure you understand that the time units are arbitrary – there is no need to use seconds/days/years to make the equations work!

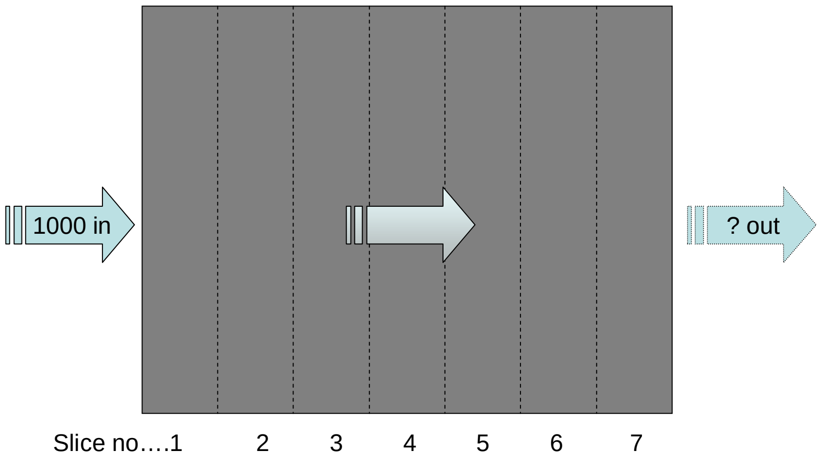
In this first example, we used an exponential to describe a situation when a population is subject to a *constant probability per unit time*.

Example 2

Let's now look at a different situation - the absorption of X-ray photons in a block of material.

For the sake of simplicity, we will assume that the photons will interact with the material in just one way: photoelectric absorption. In a photoelectric absorption event, the X-ray is completely absorbed, transferring all its energy to an electron in the absorber material. The probability of a photoelectric interaction depends only on the photon energy and the number of electrons which the photon encounters, so the *probability is constant per unit distance* while the photon moves through a material of constant density.

Consider a beam of photons travelling through a block of lead. To model the photon interactions, we divide the lead into a number of virtual slices, and calculate the number of photons that interact in each slice.



Consider how a population of 1000 photons survives its passage through the slices of lead.

Assume first that each photon has a fairly high probability of absorption of 0.4 per slice; this is a reasonable model for low energy X-rays

Slice number	Photons absorbed	Photons surviving
1	400	600
2	240	360
3	144	218
4	88	130
5	52	78
6	29	49
7	20	29

What are the key features of the absorption?
Where are most of the photons absorbed?

What fraction of the original population has passed completely through the lead.

Almost all the photons are absorbed, as we would expect for X-rays in lead.
Only 3% of the original photons survive.
Half the photons are lost in the first two slices, 80% in the first three, yet some still survive…

Now repeat this calculation, but now using a probability of absorption of 0.05 (or 1 in 20) per slice; this is a more appropriate model for higher energy gamma-rays.

Slice number	Photons absorbed	Photons surviving
1	50	950
2	48	902
3	45	857
4	43	814
5	41	773
6	39	734
7	37	697

Again, what are the key features of the absorption?

Now the number absorbed is almost constant as a function of slice number (it's exponential, but a very shallow one), and almost 70% or the photons penetrate the lead.

Make two plots which allow you to compare the two cases. On the first plot, plot the number of photons absorbed as a function of slice number (i.e. distance). On the second, plot the number of photons surviving as a function of slice number.

What is the average distance a photon penetrates into the lead for the high-attenuation and low-attenuation cases?

This illustrates an interesting point – it is not simple to define the 'range' of a photon being absorbed in this way. The mean free path can be approximated by 1/lambda

Photon absorption is usually modelled by an exponential (how did you guess!) of the form:

$$I = I_0 e^{-\mu x}$$

Where I is the penetration flux resulting from a initial flux of I_0 passing through material of thickness x, and a constant μ, called the linear attenuation coefficient which defines how quickly the photons are absorbed (and is in units of cm^{-1}).

In each of the two cases above, assume each virtual slice represents 1mm of lead, and therefore calculate the value of μ for each case based on the values derived for the first slice

By inserting values for the first slice:

Case 1 … $\mu = 5.1\ cm^{-1}$

Case 2 … $\mu = 0.51\ cm^{-1}$

Verify that the final transmission of each shield (ie the number of photons exiting from slice 7) is consistent with that predicted from the model.

Case 1 … 0.4% transmitted

Case 2 … 42% transmitted

Why don't the values predicted by the model and the tabulated values match perfectly?

The empirical calculation in the table only works well if the absorption in each slice is small, as it assumes that the absorption is constant through the slice. The calculation can be made more accurate by considering more, thinner slices.

An important question to ask, in each case, how thick does the lead need to be to stop all the gamma rays?

This question is impossible – or the answer is infinity! A photon always has a chance of penetrating the lead.

This illustrates an important feature of an exponential model - it never reaches zero. So radioactive material never stops being radioactive (but it does drop to a level of activity negligible compared to the local background) and no lead shield can ever give you 100% protection from gamma-rays!

Some more mathematics:

We now have two examples of where an exponential function describes a physical situation very well; in the first case it was describing a population is subject to a *constant probability per unit time,* in the second a population is subject to a *constant probability per unit distance.* In fact the reason an exponential works so well in each case is that it is the correct solution to the differential equation describing what happens to the population.

For the radioactive decay, the change in population per unit time is simply the population N, multiplied by the probability of decay for each nucleus. Writing that mathematically, we have:

$$\frac{dN}{dt} = -\lambda N$$

Solving this differential equation gives the familiar formula for radioactive decay:

$$I = I_0 e^{-\lambda t}$$

Further examples of exponentials:

Exponentials occur throughout physics, normally as a result of a physical situation which can be described as a *constant probability per unit something*. Can you say how these other exponential behaviours occur?

- Boltzmann distribution
- Discharge of a capacitor
- Terminal velocity

Further reading:

In example 2, we assumed that photoelectric absorption was the only way that X-ray photons could be absorbed in material. Other mechanisms include Rayleigh scattering, Compton scattering and Pair Production. Read about them, and understand how the probabilities for each interaction combine to give an overall probability.

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